COMPARISONS OF SEQUENTIAL PROCEDURES FOR SELECTING THE BEST BINOMIAL POPULATION

DAVID HOEL
NATIONAL INSTITUTE OF ENVIRONMENTAL HEALTH SCIENCES
and
MILTON SOBEL
UNIVERSITY OF MINNESOTA

1. Introduction

Recently the problem of selecting the best one of several binomial populations has been studied from the point of view of different sampling rules. In this paper, we compare some sequential procedures with and without early elimination. The main breakdown is between those using the cyclic play the winner (PWC) sampling rule and those using the vector at a time (VT) sampling rule.

The PWC rule orders the k given populations at random at the outset and uses this ordering in a cyclic manner. After each success, we sample from the same population; after each failure, we switch to the next population in the ordering scheme. After the kth population, we complete the cycle by going back to the first population.

The VT rule consists of taking k tuple observations, one component from each population. In a variation of this, the cyclic (VTC) rule, we start as in the PWC rule by randomizing the order of the populations and then take one observation from each population using the fixed cyclic order; thus, we need not complete the last vector in the VTC rule.

Both of the above rules can be modified as follows. Let the order of the populations sampled be $\pi_1, \pi_2, \cdots, \pi_k$. From the beginning of sampling π_1 to the end of sampling π_k, we have gone through one complete sampling cycle. Our new modification is to reorder the k populations after each complete sampling cycle; this reordering *can* depend on the observed results. We denote such a modification of the PWC and VTC rules by PWO and VTO, respectively.

Several papers dealing with the PW and VT sampling rules [6], [9], [12], and [13] consider termination rules based on a fixed sample size or on inverse sampling, that is, we sample until at least one population reaches a fixed number of

The work of Milton Sobel was supported by NSF Grant GP-11021 at the University of Minnesota.

successes. In [10], $k = 2$ and the termination rule is based on the difference of the numbers of successes. The monograph [1] deals mainly with VT sampling and a stopping rule based on likelihood ratios. (A summary of the above work can be found in [11].) Paulson [7], [8] has brought in early elimination techniques, which (except for [1]) is not a feature of the above references; some discussion of elimination procedures does appear in Chapter 9 of [1].

In this paper, we introduce a new procedure that combines the likelihood approach to the stopping rule with the PWC sampling rule. In Section 4, we derive an extension of the rule in [10] to the case of k populations, which contains the feature of early elimination. Empirical results for the PWO sampling rules are obtained and analyzed for the inverse sampling rule in Section 6. Sections 3 and 4 deal with the PW sampling while Sections 5 and 7 are partly concerned with VT sampling. In Section 5 sequential techniques developed in [3] are applied to Wald's sequential double dichotomy formulation [14], producing a binomial selection procedure with VT sampling and early elimination features.

In Section 7, we briefly describe two other VT rules, originally given in [1] and [7]. Finally, we present empirical results for all of the above procedures in Sections 6 and 7 and make appropriate comparisons.

2. Notation, definition, and requirement

Let p_i denote the single trial success for population π_i and let $q_i = 1 - p_i$, $i = 1, 2, \cdots, k$. The ordered p values are denoted by $p_{[1]} \leqq p_{[2]} \leqq \cdots \leqq p_{[k]}$. For $p_{[k]} > p_{[k-1]}$, a correct selection (CS) is defined as the selection of the population associated with $p_{[k]}$; for equality, either selection is correct. Let Δ denote the value of $p_{[k]} - p_{[k-1]}$. A procedure R is said to satisfy the (Δ^*, P^*) probability (of a correct selection) requirement if

$$P\{\text{CS}|R\} \geqq P^* \quad \text{whenever} \quad \Delta \geqq \Delta^*; \tag{2.1}$$

here Δ^* (with $0 < \Delta^* < 1$) and P^* (with $1/k < P^* < 1$) are preassigned constants. All the procedures discussed in this paper satisfy this common requirement (2.1).

Let N_i denote the sample size taken from π_i, and let N denote the sum over i of these sample sizes, $i = 1, 2, \cdots, k$. Let $N_{(i)}$ denote the sample size from the population associated with $p_{[i]}$, $i = 1, 2, \cdots, k$. Then we define our loss function by

$$L = \sum_{i=1}^{k} (p_{[k]} - p_{[i]})N_{(i)} \tag{2.2}$$

and the corresponding risk function by

$$\text{risk} = \sum_{i=1}^{k} (p_{[k]} - p_{[i]})E\{N_{(i)}\}. \tag{2.3}$$

In the applications that we have in mind, the one dealing with clinical trials is uppermost, where p_i denotes the probability of a cure using treatment i. In this application, the primary concern is to reduce the use of poorer treatments. The risk function (2.3) represents the expected number of failures that could have been avoided if we had known beforehand which treatment (or population) is best.

In addition to the above risk function, we are also interested in reducing the expected total number of observations $E\{N|R\}$ for the procedure R.

3. Likelihood procedure for play the winner sampling

In this section, we consider the likelihood procedure that is appropriate for play the winner sampling. The case of general k is considered in Section 3.1 and this is specialized to $k = 2$ in Section 3.2.

3.1. *Likelihood rule for* PW *sampling with general k.* A likelihood rule based on PW sampling and without early elimination can be developed in a manner similar to that given in [1]. We describe this procedure for $k = 3$ and specialize to $k = 2$ in Section 3.2; the generalization to arbitrary k is a straightforward extension of the case $k = 3$.

Let $S_1 \leqq S_2 \leqq S_3$ denote the current number of successes from the three populations and let F_i represent the current number of failures from the population associated with S_i, $i = 1, 2, 3$. If $S_3 = S_2$ $(> S_1)$, we associate S_3 with the smaller of the two corresponding F values; similarly for $S_1 = S_2 = S_3$, we assign S_3 to the one with the smallest F value. If it is still not determined, then we use randomization. However, it is shown below that for $P^* \geqq \frac{1}{2}$ our rule never terminates sampling when randomization is used. Let $p_{[1]} \leqq p_{[2]} \leqq p_{[3]}$ be the ordered (unknown) probabilities of success on a single trial.

The method, based on the techniques in [1], is to write the most likely of the three possible assignments of the pair (S_3, F_3) with the ordered p values and to stop sampling when the minimum (over that part of the parameter space for which $p_{[3]} - p_{[2]} \geqq \Delta^*$) of the corresponding likelihood ratio is at least P^*. More specifically, let the likelihood $L(\alpha, \beta, \gamma)$ be defined by

$$L(\alpha, \beta, \gamma) = p_{[1]}^{S_\alpha}(1 - p_{[1]})^{F_\alpha} p_{[2]}^{S_\beta}(1 - p_{[2]})^{F_\beta} p_{[3]}^{S_\gamma}(1 - p_{[3]})^{F_\gamma}, \tag{3.1}$$

where (α, β, γ) is a permutation of $(1, 2, 3)$. Let the likelihood ratio $\mathfrak{L}(3)$ be defined by

$$\mathfrak{L}(3) = \frac{L(1, 2, 3) + L(2, 1, 3)}{\Sigma L(\alpha, \beta, \gamma)}, \tag{3.2}$$

where the sum is over all $3! = 6$ possible permutations. This likelihood ratio $\mathfrak{L}(3)$ associates S_3 with $p_{[3]}$ and $\mathfrak{L}(j)$ is defined similarly to (3.2) and associates S_j with $p_{[3]}$, $j = 1, 2$. It is a basic result in [1] (pp. 17 and 18) that if any procedure R has the property at stopping time that

$$\max_j \min \mathfrak{L}(j) \geqq P^*, \tag{3.3}$$

where the minimum is over all points in the parameter space for which $p_{[3]} - p_{[2]} \geqq \Delta^*$, then R must satisfy the P^* condition (2.1). It is believed, also shown in [1] (part 1 of Theorem 6.1.1), that this minimum in (3.3) is generally attained at some generalized favorable configuration (GLF) in which

$$p_{[1]} = p_{[2]} = p - \Delta^*, \tag{3.4}$$

where we now use p to designate $p_{[3]}$ (this has not been demonstrated). Assuming that $\mathcal{L}(3)$ yields the maximum in (3.3) at termination (which is also not yet shown), it follows by straightforward algebra that we can write (3.3) in the form

$$\max_{\Delta^* \leqq p \leqq 1} \left\{ \frac{L(3, 2, 1) + L(2, 3, 1) + L(1, 3, 2) + L(3, 1, 2)}{L(1, 2, 3) + L(2, 1, 3)} \right\} \leqq \frac{1 - P^*}{P^*}. \tag{3.5}$$

Using (3.1), we obtain for (3.5) the explicit form

$$\max_{\Delta^* \leqq p \leqq 1} \left\{ \left(\frac{p - \Delta^*}{p}\right)^{T_1} \left(\frac{1 - p}{1 - p + \Delta^*}\right)^{U_1} + \left(\frac{p - \Delta^*}{p}\right)^{T_2} \left(\frac{1 - p}{1 - p + \Delta^*}\right)^{U_2} \right\} \leqq \frac{1 - P^*}{P^*}, \tag{3.6}$$

where $T_i = S_3 - S_i$ and $U_i = F_i - F_3$, $i = 1, 2$. For the case of VT sampling, we note that $F_j - F_3 = S_3 - S_j$, $j = 1, 2$ and the stopping rule (3.6) reduces to that given in [1] and in Section 7 below. For PW sampling the above does not hold and we note that $F_j - F_3$ can only take the values -1, 0, and $+1$. If $F_j - F_3 = -1$ for either $j = 1$ *or* $j = 2$, then the left side of (3.6) clearly tends to ∞, and hence the inequality cannot be satisfied. Thus, we can state the following stopping rule.

Stopping rule for procedure R_{LPW}: stop sampling as soon as $F_3 \leqq \min(F_1, F_2)$ and (3.6) holds.

The terminal decision rule is to select the population associated with S_3. If $S_3 = S_2$ and $F_3 = F_2$ at stopping time, then we randomize between these two populations with probability $\frac{1}{2}$ for each. Since $P^* > \frac{1}{3}$, we cannot terminate with both $S_3 = S_2 = S_1$ and $F_3 = F_2 = F_1$. Moreover, for $P^* \geqq \frac{1}{2}$ the value of $(1 - P^*)/P^* \leqq 1$ and the inequality (3.6) cannot hold if $S_3 = S_2$ and $F_3 = F_2$. Hence, we will never have to randomize in our termination rule when $P^* \geqq \frac{1}{2}$.

It should be noted that the procedure R_{LPW} is carried out by computing the maximum in (3.6) after every single observation and this can be tedious. However, it is possible to use (3.6) to construct a set of stopping points for any given P^*. This set turns out to be fairly small and thus becomes a convenient method of describing the entire stopping rule. Illustrations of such stopping sets are given in Table I for $\Delta^* = 0.1, 0.2$ and $P^* = 0.75, 0.90, 0.95, 0.99$. For example, for the pair ($\Delta^* = 0.2$, $P^* = 0.90$) there are 11 pairs of stopping points given in Table I. If $F_1 = F_2 = F_3$, then sampling terminates as soon as $T_2 = S_3 - S_2 \geqq 10$ and $T_1 = S_3 - S_1 \geqq 26$ *or* as soon as $T_2 \geqq 11$ and $T_1 \geqq 17$ *or*, and so forth.

A conservative variation of the above rule replaces $(1 - p)/(1 - p + \Delta^*)$ in (3.6) by its upper bound 1 when $F_3 \leqq \min(F_1, F_2)$ and we obtain the following stopping rule.

TABLE I

STOPPING POINTS FOR THE SEQUENTIAL LIKELIHOOD PROCEDURE
R_{LPW} WITH $k = 3$ POPULATIONS
$T_1 = S_3 - S_1$, $T_2 = S_3 - S_2$ as defined in the text.
Note: some numerical results for these procedures are given in Tables V, VI, and VII.

	$\Delta^* = 0.1$							
	$F_1 = F_2 = F_3$		$F_1 = F_2 = F_3 + 1$		$F_2 = F_3 = F_1 - 1$		$F_1 = F_3 = F_2 - 1$	
P^*	T_1	T_2	T_1	T_2	T_1	T_2	T_1	T_2
0.75	18	17	11	10	13	13	13	13
	19	16	13	9	15	12	14	11
	20	15	14	8	20	11	15	10
	22	14	17	7			16	9
	25	13	23	6			18	8
	29	12					20	7
	38	11					26	6
0.90	28	27	20	19	23	23	23	21
	30	26	21	18	24	22	24	20
	31	25	22	17	30	21	25	19
	33	24	24	16			26	18
	37	23	28	15			27	17
	42	22	36	14			29	16
	61	21					32	15
							40	14
0.95	35	35	26	25	30	29	30	27
	36	34	27	24	37	28	31	26
	37	33	29	23			32	25
	38	32	31	22			33	24
	41	31	34	21			34	23
	44	30	42	20			36	22
	50	29					39	21
	78	28					47	20
0.99	51	50	40	39	45	44	45	42
	52	49	41	38			46	40
	54	48	42	37			47	39
	56	47	44	36			48	38
	58	46	47	35			49	37
	63	45	53	34			51	36
	75	44					54	35
							59	34

	$\Delta^* = 0.2$							
	$F_1 = F_2 = F_3$		$F_1 = F_2 = F_3 + 1$		$F_2 = F_3 = F_1 - 1$		$F_1 = F_3 = F_2 - 1$	
P^*	T_1	T_2	T_1	T_2	T_1	T_2	T_1	T_2
0.75	9	8	4	4	6	6	6	5
	10	7	6	3	9	5	7	4
	12	6	10	2			8	3
	24	5					11	2
0.90	13	13	8	8	11	10	11	8
	15	12	9	7			12	7
	17	11	11	6			14	6
	26	10						
0.95	17	16	11	10	14	14	14	11
	19	15	13	9			15	10
	22	14	20	8			17	9
							23	8
0.99	24	24	17	17	21	21	21	18
	25	23	18	16			22	17
	27	22	20	15			23	16
	32	21					24	15

Stopping rule for the conservative likelihood procedure R'_{LPW}: stop sampling as soon as $F_3 \leqq \min(F_1, F_2)$ and

$$(1 - \Delta^*)^{T_1} + (1 - \Delta^*)^{T_2} \leqq \frac{1 - P^*}{P^*}. \tag{3.7}$$

It will be seen in Section 7 that when $P^* = 0.95$ and $\Delta^* = 0.2$, this conservative rule roughly causes a 20 per cent increase in the total expected number of observations, $E\{N\}$ above that for the procedure R_{LPW}.

3.2. *Likelihood rule for* PW *sampling with* $k = 2$. The special case $k = 2$ is of particular interest because we can make the procedure more explicit and because we can make comparisons with other procedures already studied, for example, the procedure R_{PW} in [10]. The derivation in Section 3.1 above gives for $k = 2$ the following stopping rule.

Stopping rule for $R_{\text{LPW}}(k = 2)$: stop as soon as $F_2 \leqq F_1$ (that is, $F_1 - F_2 = 0$ or 1) *and*

$$\max_{\Delta^* \leqq p \leqq 1} \left\{ \left(\frac{p - \Delta^*}{p} \right)^{S_2 - S_1} \left(\frac{1 - p}{1 - p + \Delta^*} \right)^{F_1 - F_2} \right\} \leqq \frac{1 - P^*}{P^*}. \tag{3.8}$$

After randomization, let I denote the population that we sample from first and II the other population. Then $F_1 = F_2$ in (3.8) when we are sampling from I and $F_1 = F_2 + 1$ in (3.8) when we are sampling from II and II has more successes. Hence, we stop and select I as soon as $S_2 - S_1 = t$ (if this happens before another equality below), where $t > 0$ is the smallest integer equal to or greater than the solution of

$$t = \frac{\log\left(\frac{1 - P^*}{P^*}\right)}{\log(1 - \Delta^*)}. \tag{3.9}$$

We stop and select II as soon as $S_2 - S_1 = s$ (if this happens first), where $s > 0$ is the smallest integer for which

$$\max_{\Delta^* \leqq p \leqq 1} \left\{ \left(\frac{p - \Delta^*}{p} \right)^{s} \left(\frac{1 - p}{1 - p + \Delta^*} \right) \right\} \leqq \frac{1 - P^*}{P^*}. \tag{3.10}$$

It is easily seen that $t \geqq s$ and that we will only select a population after getting a success from that same population. This differs from the procedure R_{PW} in [10] only in that we allow $t \geqq s$ and in [10] only $t = s$ is considered (see Table II).

Using the recursion formula method given in [10], we can now derive an exact expression for the $P\{\text{CS}\}$, $E\{N\}$, and the expected number of observations $E\{N_B\}$ on the poorer treatment. This will be done for arbitrary positive s and t and, as a special case, we can then set s and t equal to the values obtained above by the likelihood approach. Let p (respectively, p') be associated with population I (respectively, II), let $NT = I$ mean that the next trial is on population I, let (s, t) denote the stopping points, and define

$$\begin{aligned} P_n &= P_n(s, t) = P\{I \text{ is selected} | S_I - S_{II} = n, NT = I, (s, t)\}, \\ Q_n &= Q_n(s, t) = P\{I \text{ is selected} | S_I - S_{II} = n, NT = II, (s, t)\}. \end{aligned} \tag{3.11}$$

TABLE II

STOPPING POINTS FOR THE SEQUENTIAL LIKELIHOOD PROCEDURE R_{LPW} WITH $k = 2$ POPULATIONS

P^*	$\Delta^* = 0.1$		$\Delta^* = 0.2$	
	$F_1 = F_2$ $S_2 - S_1$	$F_1 = F_2 + 1$ $S_2 - S_1$	$F_1 = F_2$ $S_2 - S_1$	$F_1 = F_2 + 1$ $S_2 - S_1$
0.75	11	6	5	2
0.90	21	14	10	6
0.95	28	20	14	8
0.99	44	34	21	15

THE COMMON VALUES OF r REQUIRED BY THE PROCEDURE R_{PW} OF [10]
Use 7 and 8 with probability (weight) 0.679 and 0.321, respectively, in order to obtain a PCS of exactly $P^* = 0.75$ with $\Delta^* = 0.1$.

P^*	$\Delta^* = 0.1$	$\Delta^* = 0.2$
0.75	7.32	3.19
0.90	16.45	7.38
0.95	22.96	10.44
0.99	37.82	17.56

Then the PW sampling scheme, conditional on I being the better population leads to the recursion

$$P_n = pP_{n+1} + qQ_n, \qquad Q_n = p'Q_{n-1} + q'P_n, \tag{3.12}$$

with boundary conditions $P_t = 1$ and $Q_{-s} = 0$.

From (3.12), we find that

$$P_n(s, t) = \frac{q' - q\lambda^{s+n}}{q' - q\lambda^{s+t}}; \qquad Q_n(s, t) = \frac{q'(1 - \lambda^{s+n})}{q' - q\lambda^{s+t}}, \tag{3.13}$$

where $\lambda = p'/p \leqq 1$. Setting $n = 0$, we obtain the conditional PCS $= P_0(s, t)$ given that I is the better population. For the same problem, we define the dual expressions

$$P'_n = P'_n(s, t) = P\{II \text{ is selected} | S_{II} - S_I = n, NT = II, (s, t)\},$$
$$Q'_n = Q'_n(s, t) = P\{II \text{ is selected} | S_{II} - S_I = n, NT = I, (s, t)\}, \tag{3.14}$$

and let p (respectively, p') be associated with II (respectively, I). Then we find that the recursive scheme is exactly as in (3.12) with the new boundary conditions $P'_s = 1$ and $Q'_{-t} = 0$, which differ from the above only in that s and t are interchanged. It follows that the conditional PCS given that II is the better population is $Q'_0(s, t)$ and this is obtained from (3.13) by merely interchanging s and t, that is, $Q'_0(s, t) = Q_0(t, s)$. Hence, from these two conditional PCS results, we obtain

$$P\{\mathrm{CS}|R_{\mathrm{LPW}}\} = \frac{P_0(s,t) + Q_0(t,s)}{2} = \frac{q' - \frac{1}{2}(q\lambda^s + q'\lambda^t)}{q' - q\lambda^{s+t}}. \tag{3.15}$$

It is easily seen that for the three extreme cases $p' \to 0$, $p' \to p > 0$, and $p \to 1$ we obtain from (3.15), $P(\mathrm{CS}) = 1$, $\frac{1}{2}$, and $1 - (p')^t/2$, respectively.

Using an analogous method to obtain $E\{N_B\}$, the number of observations on the poorer population, we define

$$\begin{aligned} U_n &= U_n(s,t) = E\{N_{II}|S_I - S_{II} = n,\ NT = I,\ (s,t)\}, \\ V_n &= V_n(s,t) = E\{N_{II}|S_I - S_{II} = n,\ NT = II,\ (s,t)\}, \end{aligned} \tag{3.16}$$

and associate p with population I. Then the PW sampling scheme leads to the recursion

$$\begin{aligned} U_n &= pU_{n+1} + qV_n, \\ V_n &= p'V_{n-1} + q'U_n + 1, \end{aligned} \tag{3.17}$$

with boundary conditions $U_t = 0 = V_{-s}$. It can be shown (and it is sufficient to verify) that

$$\begin{aligned} U_n(s,t) &= \frac{q(t-n)}{p(1-\lambda)} - \frac{q[p + q(s+t)]\lambda^s(\lambda^n - \lambda^t)}{p(1-\lambda)(q' - q\lambda^{s+t})}, \\ V_n(s,t) &= \frac{p + q(t-n)}{p(1-\lambda)} - \frac{[p + q(s+t)]\lambda^s(q'\lambda^n - q\lambda^t)}{p(1-\lambda)(q' - q\lambda^{s+t})}. \end{aligned} \tag{3.18}$$

The conditional $E\{N_B\}$ given that I is the better population is $V_0(s,t)$. We again define new quantities dual to (3.16) by writing

$$\begin{aligned} U_n' &= U_n'(s,t) = E\{N_I|S_{II} - S_I = n,\ NT = II,\ (s,t)\}, \\ V_n' &= V_n'(s,t) = E\{N_I|S_{II} - S_I = n,\ NT = I,\ (s,t)\}, \end{aligned} \tag{3.19}$$

and letting p be associated with population II. Then we get the same recursion scheme as in (3.17) with the new boundary conditions $U_s' = 0 = V_{-t}'$, so that we need only interchange s and t in (3.18) to solve for (3.19). Hence, the conditional $E\{N_B\}$ given that II is the better population is $V_0'(s,t) = V_0(t,s)$. Hence, from (3.18),

$$E\{N_B|R_{\mathrm{LPW}}\} = \frac{U_0(s,t) + V_0(t,s)}{2} = \frac{[p + q(s+t)](1 - \lambda^t)(q' - q\lambda^s)}{2p(1-\lambda)(q' - q\lambda^{s+t})}. \tag{3.20}$$

Similarly, to find $E\{N_A\}$, we replace N_{II} by N_I in (3.16) and obtain, in place of (3.17),

$$\begin{aligned} \tilde{U}_n &= p\tilde{U}_{n+1} + q\tilde{V}_n + 1, \\ \tilde{V}_n &= p'\tilde{V}_{n-1} + q'\tilde{U}_n, \end{aligned} \tag{3.21}$$

with boundary conditions $\tilde{U}_t = 0 = \tilde{V}_{-s}$. The solution of this set is

$$\begin{aligned} \tilde{U}_n(s,t) &= \frac{q'(t-n)}{p(1-\lambda)} - \frac{q[p' + q'(s+t)]\lambda^s(\lambda^n - \lambda^t)}{p(1-\lambda)(q' - q\lambda^{s+t})}, \\ \tilde{V}_n(s,t) &= \frac{p' + q'(t-n)}{p(1-\lambda)} - \frac{[p' + q'(s+t)]\lambda^s(q'\lambda^n - q\lambda^t)}{p(1-\lambda)(q' - q\lambda^{s+t})}. \end{aligned} \tag{3.22}$$

Again, we set up the dual quantities

$$\begin{aligned} \tilde{U}'_n &= \tilde{U}'_n(s, t) = E\{N_{II} | S_{II} - S_I = n, NT = II, (s, t)\}, \\ \tilde{V}'_n &= \tilde{V}'_n(s, t) = E\{N_{II} | S_{II} - S_I = n, NT = I, (s, t)\}, \end{aligned} \tag{3.23}$$

and let p be associated with population II. Then the recursion is the same as in (3.21) with the boundary conditions $\tilde{U}'_s = 0 = \tilde{V}'_{-t}$, so that we need only interchange s and t in (3.22). Hence, the conditional $E\{N_A\}$ given that II is the better population is $\tilde{V}'_0(s, t) = \tilde{V}(t, s)$. It follows from (3.22) that

$$E\{N_A | R_{\text{LPW}}\} = \frac{U_0(s, t) + V_0(t, s)}{2} = \frac{[p' + q'(s + t)](q' - q\lambda^s)(1 - \lambda^t)}{2p(1 - \lambda)(q' - q\lambda^{s+t})}. \tag{3.24}$$

Adding $E\{N_A\}$ and $E\{N_B\}$, gives

$$E\{N | R_{\text{LPW}}\} = \left(\frac{\bar{p} + q(s + t)}{p}\right)\left(\frac{1 - \lambda^t}{1 - \lambda}\right)\left(\frac{q' - q\lambda^s}{q' - q\lambda^{s+t}}\right), \tag{3.25}$$

where $\bar{p} = (p + p')/2$ and $\bar{q} = 1 - \bar{p}$.

For the case $p = p'$, we take the limits in (3.20), (3.24), and (3.25) as $p' \to p$ and obtain

$$E\{N_B | R_{\text{LPW}}\} = E\{N_A | R_{\text{LPW}}\} = \tfrac{1}{2}E\{N | R_{\text{LPW}}\} = \frac{t(p + qs)}{2p}, \tag{3.26}$$

which is comparable with $r(p + qr)/2p$ obtained in (2.11) in [11] for the procedure R_{PW} with $s = t\ (= r)$. If, as is usually the case, we have $t > r > s$ and $st < r^2$, then each of these three expectations is smaller under R_{PW} for q close to zero and each is smaller under R_{LPW} for p close to zero. Thus, neither of these procedures can be uniformly better than the other, that is, throughout the parameter space. Since t in (3.9) is asymptotically ($\Delta^* \to 0$) like r in (2.13) of [11], it follows that the same lack of a uniform result holds in comparing R_{LPW} and the vector at a time procedure R_{VT}, that is, $E\{N | R_{\text{VT}}\}$ is smaller for $p \to 0$ and $E\{N | R_{\text{LPW}}\}$ is smaller for $p \to 1$.

To illustrate the results of procedure R_{LPW} and compare them with the procedure R_{PW} in [10], we consider the pair ($P^* = 0.95$, $\Delta^* = 0.2$) and put the results in tabular form. For the procedure R_{PW}, we need to randomize between $r = 10$ (with probability 0.555) and $r = 11$ (with probability 0.445); this achieves the P^* value $0.555(0.945) + 0.445(0.956) = 0.950$ in the LF configuration. For the procedure R_{LPW}, we randomize between the pair ($s = 7$, $t = 11$) with probability 0.434 and the pair ($s = 8$, $t = 12$) with probability 0.566; this achieves the P^* value $0.434(0.943) + 0.566(0.955) = 0.950$ in the LF configuration. In randomizing between these two particular pairs (7, 11) and (8, 12), for procedure R_{LPW}, rather than other pairs, such as (7, 12) and (8, 12), our criterion was to minimize the maximum of $E\{N_B\}$, which generally occurs at $\bar{p} = \Delta^*/2$. This also seems to minimize the maximum for $E\{N\}$ and $E\{N_A\}$, which also generally occur at $\bar{p} = \Delta^*/2$.

The comparison of R_{PW} and R_{LPW} in Table III shows that the latter has a smaller $E\{N_B\}$ and $E\{N\}$ in 17 out of the 18 entries. Thus, we have effected a

TABLE III

A COMPARISON OF PROCEDURES R_{PW} AND R_{LPW} FOR $k = 2$, $P^* = 0.95$, AND $\Delta^* = 0.2$ IN THE GLF CONFIGURATION $\Delta = 0.2$
Note that randomization was used to make $P^* = 0.95$ exactly in both cases; see text for details.

$\bar{p} = \frac{p + p'}{2}$	$E\{N_B\}$		$E\{N_A\}$		$E\{N\}$	
	R_{PW}	R_{LPW}	R_{PW}	R_{LPW}	R_{PW}	R_{LPW}
0.1	42.28	38.76	52.22	47.83	94.50	86.59
0.2	37.31	34.22	47.25	43.29	84.55	77.51
0.3	32.29	29.56	42.22	38.59	74.51	68.15
0.4	27.13	24.71	36.99	33.61	64.12	58.33
0.5	21.85	19.80	31.55	28.51	53.40	48.31
0.6	16.60	15.04	26.08	23.50	42.68	38.54
0.7	11.55	10.54	20.77	18.80	32.32	29.33
0.8	6.77	6.33	15.79	14.50	22.56	20.83
0.9	2.26	2.31	11.23	10.69	13.49	13.00

fairly uniform improvement, with emphasis on the maximum value at $\bar{p} = \Delta^*/2$, although (as was expected) the improvement is not substantial anywhere. However, in the context of clinical trials even slight decreases in $E\{N_B\}$ are important.

4. An elimination procedure R_{EPW}

For $k > 2$, we define an elimination procedure which is an extension of the procedure R_{PW} defined for $k = 2$ and studied in [10]. Under R_{PW}, we stop sampling when $|s_i - s_j| = r$, where s_i is the number of successes from π_i, $i \neq j$, $i = 1, 2, j = 1, 2$. Assuming $s_i > s_j$, we then select π_i as the better population. The approximate value of r required to satisfy (2.1) is the smallest integer equal to or greater than r_2, where

$$r_2 = \frac{\log 2(1 - P^*)}{\log (1 - \Delta^*)}. \tag{4.1}$$

We extend this procedure as follows. Population π_j is eliminated if for some π_i (not yet eliminated) $s_i - s_j = r$. Let π_k be the best population. Since

$$1 - P\{\text{CS}\} \leqq \Sigma P\ \{\pi_i \text{ eliminates } \pi_k\} \leqq (k - 1)(1 - P^*), \tag{4.2}$$

it follows that

$$P\{\text{CS}\} \geqq 1 - (k - 1)(1 - P^*). \tag{4.3}$$

If we now set the right side of (4.3) equal to $\tilde{P}^*$, solve for P^*, and substitute the result in (4.1), then it is clear that the resulting procedure which uses throughout for r the smallest integer equal to or greater than

$$r_k = \frac{\log\left\{2\frac{(1 - \tilde{P}^*)}{k - 1}\right\}}{\log (1 - \Delta^*)} \tag{4.4}$$

satisfies

$$P\{\mathrm{CS}|R_{\mathrm{EPW}}\} \geqq \tilde{P}^* \quad \text{whenever} \quad \Delta \geqq \Delta^*. \tag{4.5}$$

Monte Carlo results for R_{EPW} and comparisons with other procedures can be found in Tables V, VI, and VII (below).

5. Elimination with Wald's double dichotomy

For $k \geqq 2$, we investigate the numerical results of an elimination procedure which is derived in [5] and based on general methods from [3] applied to the double dichotomy problem as formulated by Wald [14].

This procedure R_{EVT} uses the VT sampling rule and eliminates population π_j if for some π_i (not yet eliminated),

$$s_i - s_j \geqq c + dn^*, \tag{5.1}$$

where $c > 0$ and $d \leqq 0$ are predetermined constants and n^* is the number of unlike pairs from π_i and π_j (that is, observations in the same vector of the form S, F or F, S).

We now give the values of c and d that satisfy the requirement (2.1). Define τ_0 by

$$\tau_0 = \left(\frac{1 - \Delta^*}{1 + \Delta^*}\right)^2 < 1, \tag{5.2}$$

and let τ_1 denote any value such that

$$\tau_0 < \tau_1 \leqq \frac{1}{\tau_0}. \tag{5.3}$$

It is shown in [5] that by taking

$$c = \frac{2 \log\left(\frac{k-1}{1-P^*}\right)}{\log \frac{\tau_1}{\tau_0}}, \qquad d = \frac{2 \log\left(\frac{1+\tau_1}{1+\tau_0}\right)}{\log \frac{\tau_1}{\tau_0}} - 1, \tag{5.4}$$

the requirement (2.1) will be satisfied. We select $\tau_1 = 1/\tau_0$ for our Monte Carlo studies and this implies that $d = 0$, the reason being that asymptotically ($P^* \to 1$) at the generalized least favorable configuration (that is, when $p_{[1]} = p_{[2]} = \cdots = p_{[k-1]} = p_{[k]} - \Delta^*$), the risk defined in (2.3) is minimized for this value of τ_1 (see [5]).

The use of $d = 0$ above also provides us with the analogous elimination procedure for extending the procedure R_{VT} in [10] to $k > 2$ in the same way that we extended R_{PW} in Section 4.

6. Comparisons of several play the winner procedures for $k = 2$

In this section, our aim is to make a comparison for $k = 2$ of the likelihood procedure developed in Section 3.2 with some other procedures that satisfy the

TABLE IV

EXPECTED SAMPLE SIZES FOR $k = 2$ UNDER FIVE PW PROCEDURES ($P^* = 0.95$, $\Delta^* = 0.2$)
The table gives $E\{N_B\}$, $E\{N\}$ for $\Delta = 0.2$ and $E_0\{N\}$ for $\Delta = 0$ in each cell.
For R_I and R_{IO}, use $r = 20$ and 21 with weights 0.958 and 0.042, respectively.
For R_H, use $r = 33$ and 34 with weights 0.6 and 0.4, respectively.
For R_{IT}, use $r = 20$ and 21 with weights 0.761 and 0.239, respectively.

$\bar{p} = \frac{p + p'}{2}$		R_{LPW}	R_I	R_{IO}	R_H	R_{IT}
0	$E_0\{N\}$	∞	∞	∞	65.8	40.5
	$E\{N_B\}$	38.8	80.7	80.2	26.8	20.2
0.1	$E\{N\}$	86.6	180.9	180.4	59.7	45.5
	$E_0\{N\}$	801.2	348.4	348.0	64.2	45.0
	$E\{N_B\}$	34.2	52.5	52.0	26.1	22.5
0.2	$E\{N\}$	77.5	119.3	118.8	59.0	51.4
	$E_0\{N\}$	362.6	173.0	172.5	63.2	50.6
	$E\{N_B\}$	29.6	38.2	37.5	25.2	24.9
0.3	$E\{N\}$	68.2	88.3	87.6	58.0	58.0
	$E_0\{N\}$	216.4	114.4	113.7	62.2	57.8
	$E\{N_B\}$	24.7	29.2	28.5	24.1	25.7
0.4	$E\{N\}$	58.3	69.1	68.4	56.9	61.1
	$E_0\{N\}$	143.2	84.8	84.2	61.4	65.3
	$E\{N_B\}$	19.8	22.9	22.1	22.6	22.7
0.5	$E\{N\}$	48.3	56.0	55.2	55.3	55.7
	$E_0\{N\}$	99.4	67.0	66.2	60.2	64.8
	$E\{N_B\}$	15.0	17.7	17.0	20.5	18.0
0.6	$E\{N\}$	38.5	46.0	45.2	53.2	46.6
	$E_0\{N\}$	70.0	54.8	53.9	59.0	55.3
	$E\{N_B\}$	10.5	13.5	12.5	17.5	13.6
0.7	$E\{N\}$	29.3	38.2	37.1	50.2	38.5
	$E_0\{N\}$	49.2	45.8	44.7	57.6	46.3
	$E\{N_B\}$	6.3	8.9	7.8	12.5	8.9
0.8	$E\{N\}$	20.8	30.8	29.7	45.3	31.1
	$E_0\{N\}$	33.6	38.4	37.1	55.2	38.9
	$E\{N_B\}$	2.3	2.5	2.5	2.5	2.5
0.9	$E\{N\}$	13.0	22.4	22.4	35.4	22.6
	$E_0\{N\}$	21.4	31.4	30.0	51.0	31.8
1.0	$E_0\{N\}$	11.6	20.0	20.0	33.4	20.2

same probability requirement (2.1) with $\Delta^* = 0.2$ and $P^* = 0.95$. All of our numerical entries for $k = 2$ (in Table IV) are based on exact formulas. In Table IV, we have included in each cell $E\{N_B\}$ for the LF configuration ($\Delta = 0.2$) and $E\{N\}$ for the LF ($\Delta = 0.2$) and equal parameter (EP) configuration ($\Delta = 0$). The procedures R_I and R_{IO} are inverse sampling procedures using PW sampling without and with reordering after each complete cycle, respectively.

The modified procedure R_H due to Hoel [3] uses PW sampling and scores, where the score W_A (for drug A, say) is defined by adding the successes of drug A and the failures of drug B and the termination rule is inverse sampling, that

is, stop when $\max(W_A, W_B) = r$. It has a bounded $E\{N\}$ value for $p = p' = 0$ and is therefore an improvement on R_I for small values of p.

Another procedure R_{IT}, due to Berry and Sobel [2], modifies the inverse sampling scheme by terminating the procedure either after a fixed number c of complete cycles or after one population reaches r successes, whichever occurs sooner. This procedure appears to have two preassigned constants (r, c) to specify, but both constants are used (with $r = c$) to satisfy (2.1).

Table IV (for $k = 2$) shows that for $\bar{p} = (p + p')/2 > \frac{1}{2}$ the likelihood procedure is preferable using either the risk criterion or $E\{N\}$. However, for $\bar{p} < \frac{1}{2}$ the value of $E\{N\}$ becomes infinite for all three of the procedures, R_{LPW}, R_I, and R_{IO} when $p = p'$. Procedures R_H and R_{IT}, on the other hand, have a bounded $E\{N\}$ function even for $p = p'$ and the numerical improvement for small $\bar{p}$ in Table IV, especially for $p = p'$, is very striking. It follows, as in the case of $k = 3$ in the next section, that if we had some *a priori* knowledge about the value of $\bar{p}$, we could more easily decide which of these procedures to use.

Procedure R_{IO} shows only a small improvement over procedure R_I, but it is uniform over the entire parameter space.

7. Monte Carlo simulation studies for $k = 3$

In this section, we bring together several procedures appropriate for $k = 3$ populations and make some Monte Carlo studies to compare them. The criteria for comparison are the risk function (2.3) and the expected total number of observations $E\{N\}$. The same formulation (2.1) applies to all these procedures with the common values $P^* = 0.95$ and $\Delta^* = 0.2$. Each entry in Table V corresponds to the average of the results of 1,000 experiments.

The main breakdown is between the procedures that use PW sampling and those that use VT sampling. We have included three previously published procedures. In the PW group, we include the inverse sampling procedure R_I studied in [13]. In the VT group, we include the procedure R_{BKS} which was developed in [1], by Bechhofer, Kiefer, and Sobel, for general k, but details of which are given in ([1], p. 270) only for $k = 2$. We also include in the VT group the procedure R_P due to Paulson [7]. A brief description of these procedures now follows.

Under procedure R_I, we sample cyclically from three populations with PW sampling until any one of them has r successes; it is then selected to be the best population. The Monte Carlo results for R_I given in Tables V, VI, and VII are very close to approximate values based on (3.35) and (3.37) in [13]. A table of these approximate values, not included here, gives values consistently smaller than the observed values in Table IV.

Under procedure R_{BKS}, we use vector sampling and stop as soon as

$$\left(\frac{1-\Delta^*}{1+\Delta^*}\right)^{2T_1} + \left(\frac{1-\Delta^*}{1+\Delta^*}\right)^{2T_2} \leqq \frac{1-P^*}{P^*} \tag{7.1}$$

and select the population associated with S_3. For $P^* > \frac{1}{2}$, we will not stop when $S_3 = S_2$, and hence, randomization will not be required at termination. It

should be noted that the form of this procedure in (7.1) is similar to that of the conservative procedure in (3.7), but since the latter uses PW sampling there is no direct comparability.

Under procedure R_P, we take N_{ir} observations from population π_i, $i = 1, 2, 3$, where N_{ir} is a Poisson random variable with mean J. Let s_{ir} (respectively, f_{ir}) denote the total number of successes (respectively, failures) from π_i up to and including the rth stage. Then population π_j is eliminated at stage r if for some π_i (not yet eliminated), we have

$$s_{jr} - f_{jr} \leqq s_{ir} - f_{ir} + \frac{\log \alpha}{\log \lambda} + rA(\lambda), \tag{7.2}$$

where $\alpha = (1 - P^*)/(k - 1)$,

$$A(\lambda) = \frac{J[\Delta^*(\lambda^2 - 1) - (\lambda - 1)^2]}{\lambda \log \lambda}, \tag{7.3}$$

and λ is any value between 1 and $(1 + \Delta^*)/(1 - \Delta^*)$. For our Monte Carlo studies, we use the same values for J and λ that were used in [7], namely, $J = 1$ and $\lambda = (1 + 0.75\Delta^*)/(1 - 0.75\Delta^*)$, which equals $23/17 = 1.353$ in our case. It should perhaps be remarked that the Poisson observations are not counted in computing $E\{N\}$ or the risk function.

Table V gives the empirical risk function, Table VI gives the empirical $E\{N\}$

TABLE V

RISK FOR VARIOUS PROCEDURES $k = 3$, $\Delta^* = 0.2$, AND $P^* = 0.95$
GLF configurations with 1,000 experiments per point.

	Play the winner sampling				Vector sampling		
max p_i	R_{EPW} Sobel-Weiss (elimination)	R_I Inverse sampling	R'_{LPW} Likelihood conservative	R_{LPW} Likelihood	R_{BKS} (see [1])	R_P Paulson (elimination)	R_{EVT} Wald's double dichotomy (elimination)
.20	22.82	45.47	27.57	22.83	9.95	10.14	9.95
.25	21.35	35.73	25.84	21.03	10.28	10.12	9.80
.30	19.67	29.08	23.99	20.05	10.29	10.01	9.61
.35	18.25	24.78	22.85	18.90	10.35	10.17	9.49
.40	17.25	21.34	21.58	17.60	10.25	9.95	9.44
.45	15.73	18.33	19.39	16.06	10.38	9.90	9.33
.50	14.72	16.26	18.18	14.95	10.40	10.14	9.21
.55	13.08	14.44	16.62	13.51	10.20	9.97	9.24
.60	12.02	12.63	14.61	11.89	10.23	10.01	9.32
.65	10.41	11.25	13.20	10.73	10.32	10.21	9.16
.70	8.71	9.85	11.40	9.22	10.44	10.07	9.18
.75	7.42	8.77	9.74	8.13	10.84	10.07	9.45
.80	6.13	7.41	7.85	6.36	10.70	9.95	9.52
.85	4.88	6.08	6.04	5.01	10.88	10.14	9.70
.90	3.62	4.71	4.36	3.48	10.79	10.19	9.72
.95	2.27	3.00	2.65	2.32	10.73	10.17	9.73
1.00	1.02	1.09	0.95	1.00	10.68	10.20	9.91

TABLE VI

EXPECTED TOTAL NUMBER OF OBSERVATIONS FOR VARIOUS PROCEDURES
$k = 3$, $\Delta^* = 0.2$, AND $P^* = 0.95$
GLF configurations with 1,000 experiments per point.

	Play the winner sampling				Vector sampling		
max p_i	R_{EPW} Sobel-Weiss (elimination)	R_I Inverse sampling	R'_{LPW} Likelihood conservative	R_{LPW} Likelihood	R_{BKS} (see [1])	R_P Paulson (elimination)	R_{EVT} Wald's double dichotomy (elimination)
.20	184.7	368.5	223.3	184.8	74.6	81.1	74.6
.25	177.9	291.1	210.3	171.2	77.1	81.0	76.0
.30	166.5	238.4	196.6	164.1	77.2	80.6	75.8
.35	156.4	204.1	188.2	155.5	77.6	81.7	75.8
.40	149.6	176.9	178.8	145.9	76.9	79.8	75.8
.45	138.8	153.6	162.5	134.5	77.9	79.4	75.4
.50	131.6	137.3	153.5	126.1	78.0	81.2	74.6
.55	118.3	123.1	141.8	115.1	76.5	80.0	74.8
.60	109.2	109.6	126.8	103.0	76.7	80.2	75.2
.65	97.2	99.2	116.3	94.5	77.4	82.0	74.6
.70	84.5	89.0	102.9	83.0	78.3	80.8	74.8
.75	73.4	80.9	90.3	74.7	81.3	80.6	76.9
.80	62.3	71.8	75.9	61.4	80.2	80.2	77.4
.85	52.1	63.1	62.4	51.0	81.6	81.0	79.1
.90	42.9	54.3	49.6	39.3	80.9	81.5	78.9
.95	31.4	44.3	36.5	30.2	80.4	81.7	78.8
1.00	21.2	33.4	23.7	20.2	80.1	81.3	79.7

function, and Table VII gives the estimated PCS function (or observed frequency of success).

As a group, the PW sampling procedures are different from the group of VT sampling procedures. The latter procedures have remarkably constant risk and $E\{N\}$ for varying values of max p_i, $i = 1, 2, 3$ while the former procedures appear to be monotonically decreasing with max p_i; the cross over point is about 0.65 in Table V and about 0.75 in Table VI. It follows that if we had some prior knowledge about max p_i (only), we might be better able to decide which type of sampling to use.

Among the PW sampling rules, the procedures R_{EPW} and R_{LPW} are quite similar and uniformly better than both the procedures R_I and the conservative likelihood procedure. However, from Table VI, it appears that the procedure R_{LPW} is slightly better than R_{EPW}.

For the VT sampling procedures, the procedure R_{EVT} is preferable to both R_{BKS} and R_P using either the risk or the $E\{N\}$ criterion; the differences between the latter two procedures appear to be small.

In Table VII we note, as expected, that all procedures satisfied the requirement (2.1) in all the experiments that were carried out. The PW procedures, except for conservative likelihood, came closer to the nominal value $P^* = 0.95$ than the VT procedures, and hence, were slightly more efficient in the sense that

TABLE VII

PROBABILITY OF CORRECT SELECTION FOR VARIOUS PROCEDURES
$k = 3$, $\Delta^* = 0.2$, AND $P^* = 0.95$
GLF configurations with 1,000 experiments per point.

	Play the winner sampling				Vector sampling		
max p_i	R_{EPW} Sobel-Weiss (elimination)	R_I Inverse sampling	R'_{LPW} Likelihood conservative	R_{LPW} Likelihood	R_{BKS} (see [1])	R_P Paulson (elimination)	R_{EVT} Wald's double dichotomy (elimination)
.20	1.000	1.000	1.000	1.000	1.000	.968	1.000
.25	1.000	1.000	1.000	1.000	1.000	.968	1.000
.30	1.000	1.000	1.000	1.000	.998	.966	.999
.35	1.000	1.000	1.000	1.000	.993	.973	.993
.40	1.000	.998	1.000	1.000	.984	.971	.993
.45	1.000	.991	1.000	.997	.984	.960	.989
.50	1.000	.984	1.000	.999	.980	.970	.982
.55	.996	.983	.997	.990	.957	.971	.974
.60	.991	.967	.997	.987	.961	.961	.970
.65	.994	.970	.997	.987	.964	.972	.977
.70	.991	.963	.995	.981	.973	.976	.969
.75	.972	.952	.990	.976	.983	.974	.981
.80	.970	.957	.985	.977	.983	.969	.989
.85	.967	.965	.987	.968	.994	.970	.997
.90	.968	.969	.981	.962	.999	.965	.997
.95	.952	.986	.978	.955	1.000	.966	1.000
1.00	.957	.996	.989	.969	1.000	.964	1.000

they have less "excess over the boundary". In addition, the columns of Table VII give some indication of where the least favorable configuration is for the pair ($\Delta^* = 0.2$, $P^* = 0.95$).

We wish to point out that we have not observed the expected number of stages required for termination since (1) it is not clearly defined for all procedures, and (2) it is not crucial for the application to clinical trials. It should also be pointed out that our Monte Carlo results are only for GLF configurations, where $p_{[1]} = p_{[2]} = p_{[3]} - 0.2$. In other configurations, the elimination procedures are even more preferable, because noncompeting populations can be eliminated early.

◇ ◇ ◇ ◇ ◇

The authors wish to thank Professor R. E. Bechhofer of Cornell University for his helpful comments.

REFERENCES

[1] R. E. BECHHOFER, J. KIEFER, and M. SOBEL. *Sequential Identification and Ranking Problems*, Chicago, University of Chicago Press, 1968.

[2] D. BERRY and M. SOBEL, "A truncated inverse sampling procedure for selecting the best binomial population," Department of Statistics Technical Report No. 154, University of Minnesota, 1971.

[3] D. G. HOEL, "A method for the construction of sequential selection procedures," *Ann. Math. Statist.*, Vol. 42 (1971), pp. 630–642.
[4] ———, "An inverse stopping rule for play-the-winner sampling," *J. Amer. Statist. Assoc.*, Vol. 67 (1972), pp. 148–151.
[5] ———, "A selection procedure based upon Wald's double dichotomy test," Oak Ridge National Laboratory Technical Report No. TM-3237, Oak Ridge, Tennessee, 1971.
[6] E. NEBENZAHL, "Play the winner sampling in selecting the better of two binomial populations," Department of Statistics Technical Report No. 147, University of Minnesota, 1970.
[7] E. PAULSON, "Sequential procedures for selecting the best one of several binomial populations," *Ann. Math. Statist.*, Vol. 38 (1967), pp. 117–123.
[8] ———, "A new sequential procedure for selecting the best one of k binomial populations," Abstract, *Ann. Math. Statist.*, Vol. 38 (1969), p. 1865.
[9] M. SOBEL and M. HUYETT, "Selecting the best one of several binomial populations," *Bell System Tech. J.*, Vol. 36 (1957), pp. 537–576.
[10] M. SOBEL and G. H. WEISS, "Play-the-winner for selecting the better of two binomial populations," *Biometrika*, Vol. 57 (1970), pp. 357–365.
[11] ———, "Recent results on using the play the winner sampling rule with binomial selection problems," *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley and Los Angeles, University of California Press, 1972, Vol. 1, pp. 717–736.
[12] ———, "Play-the-winner rule and inverse sampling in selecting the better of two binomial populations," *J. Amer. Statist. Assoc.*, Vol. 66 (1971), pp. 545–551.
[13] ———, "Play-the-winner rule and inverse sampling for selecting the best of $k \geqq 3$ binomial populations," *Ann. Math. Statist.*
[14] A. WALD, *Sequential Analysis*, New York, Wiley, 1947.